Scalings and asymptotics of coherent vortices in protoplanetary disks

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Gas that is transported radially inward from the outer edge of an accretion disk and onto a forming central star must be in a nearly Keplerian orbit at all radii. To do this, it must give up part of its angular momentum and energy to the ambient gas, which in turn advects angular momentum outward via a secondary flow. Here, we set up the numerical calculation for computing this flow by obtaining simplified sets of 3D, asymptotic equations that are well-posed and can be computed by the same techniques that are used for the 3D, anelastic Euler equation. The asymptotics allow an easy parameterization of the unknown equations of energy and state and boundary conditions. It is shown analytically that the required mass and angular momentum transport cannot occur if the protoplanetary disk is barotropic. However, a small baroclinicity allows it. Scale analysis shows that if 20% of the protoplanetary disk is filled with vortices, then the required transport can occur with a large enough radially inward mass flux to satisfy the astronomical observations.

1. Introduction

The traditional picture of protoplanetary accretion disks is that they are quiescent, without coherent features (Balbus & Hawley 1996). Some researchers have argued that they are laminar (unless they are well-coupled to magnetic fields so that the Balbus-Hawley instability can be invoked), despite the fact that their Reynolds numbers are greater than 10^{14}. In contrast, we believe that the disks are likely to be filled with structures, and the goal of this paper is to lay out a framework to compute them numerically. Our motivation is that we believe that long-lived vortices are the key to solving the angular-momentum transport problem in accretion disks and also to understanding the formation of planetesimals (Barranco & Marcus, this volume). Recently, calculations of two-dimensional vortices embedded within accretion disks have been published (Adams & Watkins 1995, Bracco *et al.* 1998, Godon & Livio 1999, 2000), but they were computed with the quasi-geostrophic, shallow-water, or two-dimensional Euler equation, and we argue below that none of these are valid for protoplanetary disks.

The hydrodynamics of a protoplanetary accretion disk are governed by the Euler equation (ignoring viscosity), the continuity equation, an energy equation, and an equation of state, along with appropriate boundary and initial conditions. The equations are difficult to solve numerically because: (1) There are two very large terms present in the equations – centrifugal force and radial gravity. They nearly cancel and their small remainder governs the physics of the coherent features. (2) There are wide ranges of length and time scales which demand high resolution and small time steps in numerical computations.

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(3) The energy equation is not known (depending upon location, the gas could be optically thick or thin). (4) The boundary and initial conditions are not well known because the disk is the end-product of a collapsing, spinning gas cloud, and how that forms the disk and continues to feed energy and matter in and out of it are not known. Without knowledge of the energy equation and equation of state, it would seem hopeless to try to compute solutions; however, we shall show that with a judicious choice of asymptotic scalings, our ignorance of this information can be readily parameterized and progress can be made.

2. Physical constraints and mathematical assumptions

Radio and infrared observations indicate that protoplanetary disks are cool. In fact, they are sufficiently cold that the gas within them is not strongly ionized and cannot couple to magnetic fields. Thus we have ignored the effects of magnetic fields. Because the characteristic speed of sound c_s^2 is nearly proportional to the gas temperature for most relevant equations of state, the coolness of the disk at most locations (say, at distances from the protostar greater than 1 A.U.),

$$c_s/V_k \equiv \delta \ll 1 \tag{2.1}$$

where $V_k \equiv \sqrt{GM/r}$ is the Keplerian velocity, M is the mass of the central protostar, G is the gravitational constant, and $(r,\ \phi,\ z)$ are the cylindrical coordinates. We ignore the self-gravity of the mass in the disk and treat the protostar as if it were a point mass. Before considering a solution to the equations of motion that includes coherent features, we first examine a base flow solution (denoted by an overbar) that is steady in time and axisymmetric and in which the radial and vertical components of the velocity are zero $\bar{V}_z = \bar{V}_r = 0$. In this case the radial and z components of the Euler equation reduce to

$$\bar{V}_\phi^2/r = GMr/R^3 + (1/\bar{\rho})\partial \bar{P}/\partial r \tag{2.2}$$

and

$$0 = GMz/R^3 + (1/\bar{\rho})\partial \bar{P}/\partial z \tag{2.3}$$

where P and ρ are the pressure and density, and R is the spherical radial coordinate. Because $c_s^2 \sim \bar{P}/\bar{\rho}$, Eq. (2.3) implies that the disk is thin,

$$H/r \sim c_s/V_k = \delta \ll 1 \tag{2.4}$$

where H is both the disk thickness and the vertical scale-height of $\bar{P}$. Eqs. (2.2) and (2.3) along with the ϕ-component of the Euler equation can be written as

$$(\bar{V}_\phi^2/r)\hat{r} = \nabla\Phi + (\nabla\bar{P})/\bar{\rho} \tag{2.5}$$

where the gravitational potential is $\Phi \equiv -GM/R$. The curl of Eq. (2.5) shows that regardless of the of the form of the energy equation or equation of state, if the flow were barotropic, $\bar{V}_\phi$ is a function of r only. Eq. (2.2) (along with the assumption that the radial scale of $\bar{P}$ is not smaller than r) shows that $\bar{V}_\phi = V_k(r)\Big(1 + \mathcal{O}\left(\delta^2\right)\Big)$. Therefore we can write

$$\bar{V}_\phi(r,z) = V_k(r)\Big(1 + \delta^2 f(r) + \delta^2 g(r,z)\Big) \tag{2.6}$$

where f and g are order unity and where $g \equiv 0$ for a barotrope. Thus, although the disk can be time-dependent and contain coherent and long-lived hydrodynamic features such as vortices, the overall flow (denoted by the overbars) is nearly Keplerian.

It would appear that to make progress we either need to know the energy equation and equation of state or the functional forms of f and g; however, in the following we show that since the disk is symmetric about the mid-plane, we only need to know one dimensionless, scalar property of $\bar{V}_\phi$: $\beta \equiv -H^2(\partial^2\bar{V}_\phi/\partial z^2)/(2V_k(r_0)\gamma)$ where $\gamma \equiv L_r/r_0$, L_r is the characteristic radial length scale, *i.e.*, $r - r_0$ of any coherent feature, and where the derivative is computed at mid-plane ($z = 0$) and at the radial location r_0 defined below. Eq. (2.6) requires that $\beta\gamma \le \delta^2$.

3. Equations, scalings and asymptotic reductions in the rotating frame

In a reference frame rotating with angular velocity $\Omega \equiv \bar{V}_\phi(r = r_0, z = 0)/r_0 = \sqrt{GM/r_0^3}\Big(1 + \mathcal{O}\left(\delta^2\right)\Big)$, the Euler and continuity equations can be written

$$\frac{Dv_r}{Dt} = \frac{v_\phi^2 - \bar{v}_\phi^2}{r} + 2\Omega(v_\phi - \bar{v}_\phi) - \frac{1}{\rho}\frac{\partial P}{\partial r} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial r} \tag{3.1}$$

$$\frac{Dv_\phi}{Dt} = -\frac{1}{r\rho}\frac{\partial P}{\partial \phi} - \frac{v_r v_\phi}{r} - 2\Omega v_r \tag{3.2}$$

$$\frac{Dv_z}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial z} + \frac{1}{\bar{\rho}}\frac{\partial \bar{P}}{\partial z} \tag{3.3}$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{v}, \tag{3.4}$$

where the velocity in the rotating frame is written in lower case and where $\bar{V}_\phi$ in the rotating frame is

$$\bar{v}_\phi(r,z) = -\Big(3\Omega(r - r_0)/2\Big)\Big(1 + \mathcal{O}\left(\gamma, \delta^2\right)\Big) - \beta c_s\gamma(z/H)^2/\delta \tag{3.5}$$

in which we have dropped the z-derivatives in v_ϕ higher than second.

Our requirement in this paper that $\gamma \ll 1$ is important is the result of the following set of physical arguments. Keplerian disks are special environments. Numerical calculations by others as well as our own experiences in hypersonic flows suggest that hypersonic and supersonic waves are transients that radiate away and leave behind subsonic vortices. Although this might not always be true, we believe that our best chance of finding coherent structures is subsonic vortices. Also, from our experience in computing long-lived 2D (Marcus 1993) and 3D (Marcus 1984, Marcus & Tuckerman 1987) coherent vortices embedded in shearing flows, we have always found that the vortices are ripped apart by the shear unless their characteristic velocities are at least as large as the differential velocity of the ambient, shearing flow. These two arguments along with Eqs. (2.4) and (3.5) and the definition of γ give the scaling

$$1 \gg \langle v\rangle/c_s \sim \Omega L_r/c_s = \Omega r_0\gamma/c_s = \gamma/\delta = L_r/H \tag{3.6}$$

where $\langle v\rangle$ is the characteristic value of v_ϕ. Eq. (3.6) implies $\gamma \ll \delta$ and $H \gg L_r$. It also implies that the Rossby number $Ro \equiv \langle v\rangle/2\Omega L_r$ is order unity. This is unlike the physical conditions of the vortices embedded in the shearing, azimuthal flows on Jupiter, where the rapid rotation of the planet, compared with the shear, makes the Coriolis force dominate the inertial force. Here, they are the same order. The relation $H \gg L_r$ implies that even though the protoplanetary disks are thin, they are not "shallow" in the context of the shallow-water equations or quasi-geostrophic equations (which are derived using the assumptions that $H \ll L_r$ and $H^2 \ll L_r^2 Ro$, respectively). Thus subsonic, coherent vortices in a protoplanetary disk are not shallow, and attempting to use the shallow-water equations or quasi-geostrophic equations to compute them is inconsistent.

Defining $P \equiv \bar{P} + \tilde{P}$, $\rho \equiv \bar{\rho} + \tilde{\rho}$, and $\epsilon^2 \sim \tilde{P}/\bar{P} \sim \tilde{\rho}/\bar{\rho}$, we can write the Euler and continuity equations in Cartesian form with $v_r \to v_y$, $v_\phi \to -v_x$, $(r - r_0) \to y$, and $r_0\phi \to -x$. Keeping leading order terms to $\mathcal{O}\left(\delta^2, \epsilon^2, \gamma\right)$:

$$\frac{\partial u_x}{\partial t} = -\nabla \cdot \Big(\mathbf{u} u_x / \bar{\rho}(r_0, z)\Big) + 2\Omega u_y - \frac{\partial \tilde{P}}{\partial x} \tag{3.7}$$

$$\frac{\partial u_y}{\partial t} = -\nabla \cdot \Big(\mathbf{u} u_y / \bar{\rho}(r_0, z)\Big) - 2\Omega (u_x - \bar{u}_x) - \frac{\partial \tilde{P}}{\partial y} \tag{3.8}$$

$$\frac{\partial u_z}{\partial t} = -\nabla \cdot \Big(\mathbf{u} u_z / \bar{\rho}(r_0, z)\Big) - \frac{\partial \tilde{P}}{\partial z} - \tilde{\rho} z \Omega^2 \tag{3.9}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{3.10}$$

Here the momentum $\mathbf{u} \equiv \bar{\rho}(r_0, z)\mathbf{v}$, $\bar{u}_x \equiv \bar{\rho}(r_0, z)\bar{v}_x$, and in deriving Eq. (3.10) we assume that the time-scale is of the same order as the advective time (in the rotating frame) or slower, (*cf.*, Eq. (3.11) below). Note that acoustic and other fast waves are neglected in this approximation.

We plan to solve Eqs. (3.7)-(3.10) numerically using the standard methods for the Euler equation with an anelastic equation of state, but to see what the solutions might look like and to make analytic progress, we now examine three different asymptotic regimes. To do so, we choose units for the thermodynamic quantities, momenta, x, y, and z, such that the non-dimensionalized quantities are order unity. Only the leading order terms to $\mathcal{O}\left(\delta^2, \epsilon^2, \gamma\right)$ are retained. Although the units of x, z, and u_z will differ for the different asymptotic scalings, all three share the following (where square brackets mean "units of"):

$$[\bar{\rho}] = \bar{\rho}(r_0, 0) \qquad [\bar{P}] = [\bar{\rho}]c_s^2 \qquad [L_y] \equiv L_r = \epsilon H$$

$$[u_x] = \epsilon[\bar{\rho}]H\Omega \qquad [u_y] = \epsilon^2[\bar{\rho}]H^2\Omega/[L_x] \qquad [t] = [L_x]/\epsilon H\Omega$$

$$[\tilde{\rho}] = \epsilon^2[\bar{\rho}] \qquad [\tilde{P}] = \epsilon^2[\bar{P}], \tag{3.11}$$

where c_s is evaluated at the mid-plane of the disk at $r = r_0$. The scaling for u_x follows from requiring that the Rossby number be order unity. The length scale L_y results from this and our desire to have the Mach number $[v_x]/c_s \equiv \epsilon$. The scale of u_y is chosen by demanding that the Coriolis terms are of the same order as the pressure terms. As a consequence of this last scaling, the x- and y-components of the advective derivative also have the same order. The scaling for $\bar{P}$ follows from the definition of c_s. The scaling for the pressure deviations $\tilde{P}$ arises from the requirement that the pressure and Coriolis terms are of the same order in the y-component of the momentum equation. The scaling for $\tilde{\rho}$ comes from the requirement that the fractional changes in pressure and density are the same order. The choice of time scale comes from requiring that it is the advective time-scale. We shall see that it replaces the dynamics continuity equation with the kinematic condition that the mass flux is divergence-free. This removes a temporal degree of freedom, and so sound and supersonic waves are removed from the system of equations.

The scaling for $[L_x]$ displayed in Eq. (3.11) as well as $[L_y]$ and $[u_z]$ have been left unspecified at this point because there is some freedom in how we choose them. This freedom will lead to different physical and dynamical regimes which we will show below.

Equation (2.4), the scaling $[L_y]$, and the definition of γ show that ϵ, γ, and δ are not independent but instead satisfy

$$\epsilon \equiv \gamma/\delta \ll 1. \tag{3.12}$$

The non-dimensionalized equations can now be written in terms of the (not yet specified) constants $[L_z]$, $[L_x]$, and $[u_z]$,

$$\frac{\partial u_x}{\partial t} = -\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_x}{\bar{\rho}(r_0,z)}\right) - \frac{\partial\Big(u_z u_x/\bar{\rho}(r_0,z)\Big)}{\partial z}\left\{\frac{[u_z]}{[u_x]}\frac{[L_x]}{[L_z]}\right\} + 2u_y - \frac{\partial\tilde{P}}{\partial x} \tag{3.13}$$

$$\begin{aligned}\frac{\partial u_y}{\partial t} = &-\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_y}{\bar{\rho}(r_0,z)}\right) - \frac{\partial\Big(u_z u_y/\bar{\rho}(r_0,z)\Big)}{\partial z}\left\{\frac{[u_z]}{[u_x]}\frac{[L_x]}{[L_z]}\right\}\\ &-2(u_x - \bar{u}_x)\left\{\frac{[L_x]^2}{[L_y]^2}\right\} - \frac{\partial\tilde{P}}{\partial y}\left\{\frac{[L_x]^2}{[L_y]^2}\right\}\end{aligned} \tag{3.14}$$

$$\begin{aligned}\frac{\partial u_z}{\partial t} = &-\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_z}{\bar{\rho}(r_0,z)}\right) - \frac{\partial\Big(u_z^2/\bar{\rho}(r_0,z)\Big)}{\partial z}\left\{\frac{[u_z]}{[u_x]}\frac{[L_x]}{[L_z]}\right\}\\ &-\frac{\partial\tilde{P}}{\partial z}\left\{\frac{[u_x]}{[u_z]}\frac{[L_x]}{[L_z]}\right\} - \tilde{\rho} z\left\{\frac{[u_x]}{[u_z]}\frac{[L_x]}{H}\frac{[L_z]}{H}\right\}\end{aligned} \tag{3.15}$$

where the $\perp$ subscript means the x and y components. The non-dimensional steady state azimuthal velocity is

$$\bar{u}_x = \left(\frac{3}{2}y + \beta z^2[L_z]^2/H^2\right)\bar{\rho}(r_0,z) \tag{3.16}$$

It should be kept in mind that unlike the momenta and thermodynamic quantities, the non-dimensional z can be much greater than unity: the dimensional z is order H; the non-dimensional z is order $H/[L_z]$ which can be big (see §3.2).

Without an energy equation, it is impossible to obtain an expression for $\tilde{\rho}$ which is needed in the buoyancy terms of the equations above, so we exploit a standard method used in geophysical fluid dynamics. The $\bar{\rho}$ is sensitive to the equations of state and energy (and boundary conditions) because it represents a long-time balance within the disk of energy sources and sinks (*e.g.*, we cannot compute the $\bar{\rho}$ in the earth's atmosphere without taking into account the effects of ground heating, cloud cover, cooling, *etc.*). However, the density disturbances $\tilde{\rho}$ within the disk are created by advection of fluid parcels. If the advective time is fast compared with the thermal time (which is unknown and due to complicated physics), then $\tilde{\rho}$ is nearly equal to that of an adiabatic displacement, and if the time scales have the opposite ordering, then $\tilde{\rho}$ is approximated as an isothermal displacement. In the case where the two time scales are equal (which would be unusual since they are determined by very different dynamics), an energy equation is needed. Most previous computations of protoplanetary disks use an adiabatic approximation for $\tilde{\rho}$ and assume an ideal gas equation of state, and that will also be our starting assumption (to be modified later if it is required).

We have found three different relationships for $[L_x]$, $[L_z]$, and $[u_z]$ that are physically meaningful and yield mathematically consistent asymptotic equations. We believe them to be exhaustive and present them below.

3.1. *Columnar dynamics*

One set of asymptotic equations is obtained from Eqs. (3.13)- (3.15) by setting $[L_x] = [L_y] = \epsilon H$, $[L_z] = H$, and $[u_z] = \epsilon [u_x]$. Retaining terms to $\mathcal{O}\left(\delta^2, \epsilon^2, \gamma\right)$, we obtain

$$\frac{\partial u_x}{\partial t} = -\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_x}{\bar{\rho}(r_0, z)} \right) + 2u_y - \frac{\partial \tilde{P}}{\partial x} \tag{3.17}$$

$$\frac{\partial u_y}{\partial t} = -\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_y}{\bar{\rho}(r_0, z)} \right) - 2\Big(u_x - \bar{\rho}(r_0, z)(3y/2 + \beta z^2) \Big) - \frac{\partial \tilde{P}}{\partial y} \tag{3.18}$$

$$\frac{\partial u_z}{\partial t} = -\nabla_\perp \cdot \left(\frac{\mathbf{u}_\perp u_z}{\bar{\rho}(r_0, z)} \right) - \frac{\partial \tilde{P}}{\partial z} - \tilde{\rho} z \tag{3.19}$$

$$\nabla_\perp \cdot \mathbf{v}_\perp = 0 \tag{3.20}$$

These equations in which the 2-dimensional component of the velocity is divergence-free describe coherent features whose lengths and velocities are similar in the plane of the disk but are columnar in the sense that $[L_z] \gg [L_x] = [L_y]$. In these equations there is no contribution from the z component of the advective derivative. The fluid is decoupled from itself in the z direction. If the vertical layers in a coherent feature begin to decouple from one another, then the the vertical gradients become large, the vertical scale-height of the flow becomes much smaller than H, and the underlying scaling breaks down. The dynamics would then become governed by the more general Eqs. (3.13)-(3.15) which would tend to recouple the layers. The decoupling of the flow in z makes it easy to compute steady solutions to Eqs. (3.17)- (3.20), so these equations are particularly useful to find steady solutions, but not to explore dynamics or test stability.

3.2. *Round vortices*

The second asymptotic limit comes from setting $[L_z] = [L_x] = [L_y] = \epsilon H$, and $[u_z] = [u_x] = [u_y]$,

$$\frac{\partial u_x}{\partial t} = -\nabla \cdot \left(\frac{\mathbf{u} u_x}{\bar{\rho}(r_0, z)} \right) + 2u_y - \frac{\partial \tilde{P}}{\partial x} \tag{3.21}$$

$$\frac{\partial u_y}{\partial t} = -\nabla \cdot \left(\frac{\mathbf{u} u_y}{\bar{\rho}(r_0, z)} \right) - 2\Big((u_x - \bar{\rho}(r_0, z)(3y/2 + \epsilon^2 \beta z^2) \Big) - \frac{\partial \tilde{P}}{\partial y} \tag{3.22}$$

$$\frac{\partial u_z}{\partial t} = -\nabla \cdot \left(\frac{\mathbf{u}_\perp u_z}{\bar{\rho}(r_0, z)} \right) - \frac{\partial \tilde{P}}{\partial z} - \epsilon^2 \tilde{\rho} z \tag{3.23}$$

$$0 = \nabla \cdot \mathbf{u} \tag{3.24}$$

Note, to be consistent, we have kept the $\mathcal{O}\left(\epsilon^2\right)$ terms in the baroclinic β term in Eq. (3.22) and in the buoyancy term in in Eq. (3.23). As we stated after Eq. (3.16), $z \sim H/L_z = \epsilon^{-1}$. Thus the β term is actually order unity and the buoyancy term in Eq. (3.23) is $\mathcal{O}(\epsilon)$ and must be retained. Qualitatively speaking, this scaling is valid for nearly spherically shaped vortices.

3.3. *Elongated dynamics*

In this final set, $[L_x] = [L_z] = H$ and $[u_x] = [u_z]$, resulting in,

$$\frac{\partial u_x}{\partial t} = -\nabla \cdot \left(\frac{\mathbf{u} u_x}{\bar{\rho}(r_0, z)} \right) + 2u_y - \frac{\partial \tilde{P}}{\partial x} \tag{3.25}$$

$$0 = -2\Big(u_x - \bar{\rho}(r_0, z)(3y/2 + \beta z^2)\Big) - \frac{\partial \tilde{P}}{\partial y} \tag{3.26}$$

$$\frac{\partial u_z}{\partial t} = -\nabla \cdot \left(\frac{\mathbf{u} u_z}{\bar{\rho}(r_0, z)}\right) - \frac{\partial \tilde{P}}{\partial z} - \tilde{\rho} z \tag{3.27}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{3.28}$$

Vortices with this scaling are stretched in the azimuthal direction. Furthermore, the dynamics in the radial (y) direction is geostrophic, making the effective Rossby number in that direction zero. It can be shown that these equations, even though there is no time derivative in Eq. (3.26), have the same boundary condition requirements as the anelastic Euler equations.

4. Linear theory

In this section we examine the linear stability of the unperturbed disk using the asymptotic equations (3.25)-(3.28) derived for the elongated dynamics in §3.3. Stability (or instability) computed with these equations does not guarantee stability (or instability) when computed with the full equations, but any eigenmode computed with Eqs. (3.25)-(3.28) whose length and time scales are consistent with the those of the assumptions in §3.3 is valid. The dual purposes of this section are first to illustrate some possible dynamics of the unperturbed disk and second to show that subtle changes in boundary conditions can lead to big differences. For one choice of boundary conditions we show that the disk modes exhibit algebraic singularities at some critical time t_{sing}. However, at early times prior to t_{sing} the solutions are consistent with the assumptions used to derive the asymptotic equations. Nonetheless, the prediction of a violent instability suggests that when the full equations are used, the disk is either unstable or there are initial conditions that lead to transient modes that grow before they decay. Even if the disk is linearly stable when computed with the full equations, if the transients reach large amplitudes they could trigger a finite-amplitude instability. This suggests that we look for these transients in the numerical simulations. For the other set of boundary conditions, stability properties change. This suggests that numerical simulations will need to be computed with a variety of physically reasonable boundary conditions to understand fully the physics of the disk.

We linearize Eqs. (3.25)-(3.28) about the unperturbed disk with momentum flux $\bar{u}_x$. We choose the simplest energy equation and equation of state: the fluid has constant density. This makes $\tilde{\rho} \equiv 0$, $\nabla \cdot \mathbf{v} = 0$, and $\beta \equiv 0$. Writing perturbed quantities with a "prime" and using $\mathbf{v}$ rather than $\mathbf{u}$ as the independent variable, we obtain

$$\left(\frac{\partial}{\partial t} + \bar{v}_x \frac{\partial}{\partial x}\right) v_x' = -\frac{\partial \tilde{P}'}{\partial x} + \tfrac{1}{2} v_y' \tag{4.1}$$

$$0 = -\frac{\partial \tilde{P}'}{\partial y} - 2 v_x' \tag{4.2}$$

$$\left(\frac{\partial}{\partial t} + \bar{v}_x \frac{\partial}{\partial x}\right) v_z' = -\frac{\partial \tilde{P}'}{\partial z} \tag{4.3}$$

$$0 = \nabla \cdot \mathbf{v}' \tag{4.4}$$

We now consider two types of boundary conditions.

4.1. *Channel geometry*

Here we use the boundary conditions similar to those in an inviscid channel flow:

$$\text{(i) } v_z' = 0 \text{ at } z = \pm\zeta \qquad \text{and} \qquad \text{(ii) } v_y' = 0 \text{ at } y = \pm 1, \tag{4.5}$$

with periodic boundary conditions in x. Making each perturbation quantity have x and t dependence of $e^{i(\omega t + k_x x)}$, we combine Eqs. (4.1)-(4.4) into a single equation for $\tilde{P}'$,

$$-\left(\omega + \tfrac{3}{2}k_x y\right)^2 \frac{\partial^2 \tilde{P}'}{\partial y^2} + \frac{\partial^2 \tilde{P}'}{\partial z^2} = 0, \tag{4.6}$$

where the velocity can be expressed in terms of $\tilde{P}'$:

$$v_x' = \frac{1}{2}\frac{\partial \tilde{P}}{\partial y} \quad , \quad v_y' = -i\left(\omega + \tfrac{3}{2}k_x y\right)\frac{\partial \tilde{P}'}{\partial y} + 2ik_x \tilde{P}'. \tag{4.7}$$

Equation (4.6) is separable, so we write

$$\tilde{P}' \equiv \Psi_{m\ell}(y) Z_m(z). \tag{4.8}$$

In enforcing the boundary condition at $z = \pm\zeta$, we find that

$$Z_m(z) = \cos\Big((2m+1)\pi z / 2\zeta\Big), \tag{4.9}$$

where m is an integer. The velocities in the x and y directions and the pressure are proportional to cosines in z, and v_z' is proportional to a sine. The equation for $\Psi_{m\ell}$ is now an equi-density equation with power-law solution:

$$\Psi_{m\ell} = \left(\frac{\omega_{m\ell} + \frac{3}{2}k_x y}{\omega_{m\ell} - \frac{3}{2}k_x}\right)^{\frac{1}{2} + i\frac{1}{2}\Delta} + \left(\frac{5 + 3i\Delta}{5 - 3i\Delta}\right)\left(\frac{\omega_{m\ell} + \frac{3}{2}k_x y}{\omega_{m\ell} - \frac{3}{2}k_x}\right)^{\frac{1}{2} - i\frac{1}{2}\Delta}, \tag{4.10}$$

where

$$\Delta \equiv \left(\frac{4\pi^2(2m+1)^2}{9\zeta^2 k_x^2} - 1\right)^{\frac{1}{2}}, \tag{4.11}$$

and where the frequencies ω are labeled with two subscripts, $\omega_{m\ell}$, and satisfy the dispersion relation

$$\omega_{m\ell} = \tfrac{3}{2}k_x \coth\left(\tfrac{\ell\pi}{\Delta}\right), \qquad \text{for } 2\pi|2m+1| - 3\zeta|k_x| > 0 \tag{4.12}$$

$$\omega_{m\ell} = i\tfrac{3}{2}k_x \cot\left(\tfrac{\ell\pi}{|\Delta|}\right), \qquad \text{for } 2\pi|2m+1| - 3\zeta|k_x| < 0, \tag{4.13}$$

and where ℓ is a non-zero integer. The eigenmodes are unstable when $\omega_{m\ell}$ has a positive real part or when $2\pi|2m+1| - 3\zeta|k_x| < 0$ and $0 < \text{mod}_\pi(\ell\pi/|\Delta|) < \pi/2$ hold simultaneously.

4.2. *Sliding box coordinates*

In this section we show that the "sliding box" boundary conditions give algebraic rather than exponential behavior. We introduce the "sliding box" coordinates that were previously used in studies (Marcus & Press 1977, Rogallo 1981 and Korycansky 1992) of plane Couette and other shearing flows:

$$\hat{x} \equiv x + \frac{3}{2}yt \qquad \hat{t} \equiv t \qquad \hat{y} \equiv y \qquad \hat{z} \equiv z \tag{4.14}$$

Eqs. (4.1)-(4.4) are autonomous in the new spatial coordinates but no longer so in time. Making all of the perturbed variables proportional to $e^{i(k_y\hat{y}+k_x\hat{x})}$, the linearized Eqs. (4.1) - (4.4) in the new coordinates are:

$$\frac{\partial v_x'}{\partial \hat{t}} = -ik_x\tilde{P}' + \tfrac{1}{2}v_y' \tag{4.15}$$

$$0 = -2v_x' - i(k_y - \tfrac{3}{2}k_x\hat{t})\tilde{P}' \tag{4.16}$$

$$\frac{\partial v_x'}{\partial \hat{t}} = -\frac{\partial}{\partial \hat{z}}\tilde{P}' \tag{4.17}$$

$$0 = ik_xv_x' + i(k_y - \tfrac{3}{2}k_x\hat{t})v_y' + \frac{\partial}{\partial \hat{z}}v_z'. \tag{4.18}$$

Requiring, as before, that there be no vertical flow at $\hat{z} = \pm\zeta$ requires that $\tilde{P}'$ be proportional to $Z_m(\hat{z})$ where Z_m and m are defined as they were in Eq. (4.9). Writing $\tilde{P}' \equiv \mathcal{P}(t)Z_m(\hat{z})e^{i(k_y\hat{y}+k_x\hat{x})}$, Eqs. (4.15)-(4.18) can be combined into a single ordinary differential equation in time for $\mathcal{P}$,

$$\tfrac{9}{4}k_x^2\frac{d}{dT}\left(2T + T^2\frac{d}{dT}\right)\mathcal{P} + \frac{\pi^2(2m+1)^2}{\zeta^2}\mathcal{P} = 0, \tag{4.19}$$

where we have used the temporal coordinate

$$T \equiv= \tfrac{3}{2}k_x\hat{t} - k_y. \tag{4.20}$$

Equation (4.19) has a regular singular point at $T = 0$ which shows that the solution $\mathcal{P}$ will be algebraically unstable. $\mathcal{P}$ is given by,

$$\mathcal{P} = a\Big(\tfrac{3}{2}k_x\hat{t} - k_y\Big)^{\chi_+} + b\Big(\tfrac{3}{2}k_x\hat{t} - k_y\Big)^{\chi_-} \qquad \chi_\pm = -\frac{3\pm\nu}{2}\ ,\ \nu = \left(1 - \frac{4(2m+1)^2\pi^2}{9\zeta^2k_x^2}\right)^{\frac{1}{2}}, \tag{4.21}$$

where a and b are integration constants and depend on the initial condition.

All perturbations in which $k_yk_x > 0$ have an algebraic singularity at $t_{\rm sing} \equiv 2k_y/3k_x$ and grow in time. Modes with $k_yk_x < 0$ decay algebraically. This suggests that the unperturbed disk computed with the full equations of motion is either algebraically unstable or has transients that grow before they decay. This behavior is different from that computed with the channel boundary conditions in §4.1.

5. Discussion and conclusions

Although we have not yet numerically solved our asymptotic equations, we can draw several conclusions about their solution. First, it is almost certain that they allow coherent anti-cyclones (vortices opposite in sign to Ω). Two- and three-dimensional numerical calculations of vortices embedded in shearing flows show that if the shear and the vorticity are of the same order, they must also be the same sign; otherwise, the vortices are stretched by the ambient flow and destroyed (Marcus 1993). Moreover vortices embedded in like-signed shearing flows with Rossby numbers less than or order unity (ours are designed to be order unity by our choice of the asymptotic scalings) are very stable; small ones tend to merge together and become large; when turbulence rips a vortex apart, the fragments often merge and restore the vortex. Since the shear in a Keplerian disk is anti-cyclonic, anti-cyclones would likely be stable in protoplanetary disks. Two-dimensional simulations of near-Keplerian disks (Adams & Watkins 1995, Bracco *et al.*

1998, Godon & Livio 1999, 2000) confirm this, and we expect the stability to remain valid in 3-dimensional disks.

Moreover, because our solutions have Rossby numbers of order unity, they will be in partial geostrophic balance (*i.e.* the Coriolis force will partially balance the pressure gradient). Geostrophic balance makes anti-cyclones have relatively high pressures in their interiors or positive $\tilde{P}$ and cyclones have relatively low pressures or negative $\tilde{P}$. For isothermal or adiabatic perturbations (the types considered here), positive $\tilde{P}$ goes along with positive $\tilde{\rho}$ (see §3), which means that anti-cyclones in a protoplanetary disk correspond to mass over-densities.

Our overall picture of mass and angular momentum transport in a protoplanetary disk is that the perturbations of the in-falling mass at the outer edge of the disk create anti-cyclones, or lumps of mass over-densities that are long-lived. In future work we shall test this hypothesis numerically. However, the question still remains as to whether the lumps migrate radially inward. We can state with certainty that if the disk is barotropic they do not. This can easily be seen from Eqs. (3.7)-(3.10) (or any of our three sets of asymptotic equations.) In all cases, if $\beta = 0$, the equations are invariant under the symmetry $x \to -x$, $y \to -y$. Due to this symmetry, there is nothing to distinguish the radially inward direction from the radial outward direction (other than the geometrical curvature of the disk which is small compared to other small quantities and is ignored in a first-order asymptotic expansion). This means that if a mass lump or anti-cyclone were placed in the flow, it could not migrate radially. When the flow is baroclinic and $\beta \neq 0$, this symmetry is broken, and the vortex is free to drift radially.

An important question to answer before tackling the equations numerically is whether the secondary flow due to the vortices is large enough to transport the requisite angular momentum radially outward. The inward mass flux that forms the star (due to the inward drift of anti-cyclonic lumps in our picture and due to unspecified "turbulence" or laminar inward flow in other scenarios) also carries angular momentum inward. The secondary flow due to the vortices must compensate for this angular momentum flux which is $r^2\Omega_k\dot{M}$ where $\dot{M}$ is the radially inward mass flux and equal to the rate at which the protostar gains mass, and Ω_k is the Keplerian angular velocity. The outward flux of momentum due to secondary flows (including vortices) is approximately $2\pi r^2 H[u_y][u_x]/[\rho]Cf = fC\epsilon^2 2\pi r^2 H^3[\rho]\Omega_k^2$, where we have used the round scaling in section 3.2 to estimate the characteristic radial and azimuthal velocities of the secondary flow $[u_y]$ and $[u_x]$, C is the correlation between the radial and azimuthal components of the velocity, and f is the fraction of the disk filled with the secondary flow (vortices). Setting these two fluxes equal and using (at r equal to one A.U. - the distance from the earth to the sun) $[\rho] = 1.4 \times 10^{-9}\text{g/cm}^3$, $r = 1.5 \times 10^{13}\text{cm}$, $H = 4.5 \times 10^{11}\text{cm}$, $\Omega_k = 2\pi$ year^{-1}, and $\dot{M} = 10^{-8}$ solar masses per year (with one solar mass equal to 2.0×10^{33}g), we obtain $fC\epsilon^2 = 4 \times 10^{-3}$. We do not know the value of C *a priori*; it must be computed. However, numerical simulations of the vortices in Couette-Taylor flow in which the vortices are the main transporters of the radial angular momentum flux have $C \sim 0.1$. We set $\epsilon^2 = 0.2$. (Our *physical* assumption that robust vortices are subsonic restricts $\epsilon^2 < 1$; our *mathematical* requirement to obtain the asymptotics of the round-vortex equations requires that $\epsilon^2 \ll 1$ so that ϵ^2 could be an expansion parameter. Setting $\epsilon^2 = 0.2$ may be too conservative, and in the future it may be necessary to obtain an asymptotic expansion for the disk equations in δ and γ that does not require $\epsilon^2 \ll 1$.) With these values we obtain $f = 1/5$, meaning that if one fifth of the disk were filled with vortices, then angular momentum balance could be maintained with the observed mass accretion rates. We caution the reader that the estimates for the observed ρ and H could be incorrect by factors of 2 or more, and so could our scaling estimates for the values of $[u_y]$ and $[u_x]$. However, it is encouraging that the estimates of f are not several orders of magnitude greater than unity.

Of course, the way to interpret these results from a physical point of view is that computations (or the physics of an actual protoplanetary disk) provide the values of f, C, H, u_x, u_y, and ρ, which in turn determine the observed value of $\dot{M}$. With this in mind, our goals in computing solutions to Eqs. (3.13)-(3.15) are: (1) Start with an initial anti-cyclone embedded in a nearly-Keplerian, three-dimensional disk and determine if it is stable and long-lived. Since we shall not know *a priori* its equilibrium shape, watch it relax to its equilibrium and determine the physics of how it relaxes. (2) Determine the anti-cyclone's sensitivity to the boundary conditions of the disk. (3) Determine how a large anti-cyclone can be created when it is not initially present in the flow. Can it be created by repeated mergers of much smaller (initial) anti-cyclones or from an initial set of disturbances in $\tilde{\rho}$? Determine whether an anti-cyclone can be created in an initially undisturbed flow (*i.e.* the base disk flow written with the overbars). Is the flow linearly unstable? If the flow is stable, can an anti-cyclone be created in the base disk flow if the mass in-flow through the outer boundary condition is made time-dependent and variable in ϕ (thereby creating mass "lumps" at the outer boundary)? (4) Determine the rate at which the anti-cyclones drift radially inward as a function of baroclinicity and determine whether it is sufficient to reproduce the observed mass accretion rates.

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