1. A bead slides without friction on a rotating wire as shown in the figure below. The straight wire is oriented at constant polar angle $\theta$, rotating about a vertical axis with constant angular velocity $\Omega = \dot{\phi}$. The apparatus is in a uniform gravitational field $g$ aligned with the vertical direction.

(a) Construct the Lagrangian using the radial distance $r$ from the pivot ($z = r \cos \theta$) as the independent coordinate.

(b) Show that the condition for an equilibrium circular orbit is given by

$$r = \frac{g \cos \theta}{(\Omega \sin \theta)^2}$$

(c) Discuss the stability of the orbit against small displacements by expanding $r(t) = r_0 + \eta(t)$, where $\eta$ is a small quantity.

2. Show that if $q_i = q_i(t)$ are solutions of the Euler-Lagrange equations for a given Lagrangian $L(q_i, \dot{q}_i, t)$, then they are also solutions of a new Lagrangian

$$L'(q_i, \dot{q}_i, t) = L(q_i, \dot{q}_i, t) + \frac{df(q_i, t)}{dt}$$

where $f$ can be any arbitrary function of the generalized coordinates $q_i$.

3. **Mathematica Project:** This problem is tracked separately from other homework for grading purposes. This problem concerns the double pendulum with massless rods of length $\ell$ and equal masses $m$.

(a) Create a full, general Mathematica solution of the problem as represented in L&L problem (1), using the Lagrangian formalism. Your solution should start with the Lagrangian, and derive all equations of motions from it.
(b) Use Mathematica or a similar program to plot the motion of the free end of the double pendulum.

Plot the motion in the \((x_2, y_2)\) plane as well as in the phase space \((\theta_2, p_{\theta_2})\) plane. Your plots ought demonstrate the following set of initial conditions, all of them released from rest:

- Small initial displacement for both angles.
- Large initial displacement for the top pendulum, and small for the bottom.
- Large initial displacement for the bottom pendulum, and small for the top.
- Large initial displacement for both of the pendula.

For the final case, try to see what happens when you change your starting angle on the top pendulum slightly.

The attached examples ought to get you going. They shows the motion for the simple harmonic oscillator, and also the following system of equations with initial conditions \(x = 1, y = 1, \dot{x} = 0, \dot{y} = 0:\)

\[
\ddot{x} = \sin 5y; \ddot{y} = \cos 5x
\]

N.B. the above equation is just an example and does not actually correspond to the correct set of equations for this problem.

(c) Repeat the above problem, including derivations, for the case that the top pivot of the pendulum is rotating with uniform angular frequency \(\omega = \sqrt{g/l}\). Solve for the motion in the rest frame of the pivot. You only need to do the "small initial displacement for both angles" case.